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The mechanics of shuffle products and their siblings

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Abstract

Nous poursuivons ici le travail commenc  dans [15] en d crivant des produits de m langes d'alg bres de fonctions sp ciales (issues d' quations diff rentielles   p les simples) de plus en plus grandes. Les  tudier nous conduit   d finir une classe de produits de m lange, que nous nommons φ -shuffles. Nous  tudions cette classe d'un point de vue combinatoire, en commen ant par  tendre (sous conditions) le th or me de Radford   celle-ci, puis en construisant (toujours sous conditions) sa big bre. Nous analysons les conditions des r sultats pr cit s pour les simplifier en les rendant visible d s la d finition du produit de m lange. Nous testons enfin ces conditions sur les produits introduits en d but d'article.

We carry on the investigation initiated in [15] : we describe new shuffle products coming from some special functions and group them, along with other products encountered in the literature, in a class of products, which we name φ -shuffle products. Our paper is dedicated to a study of the latter class, from a combinatorial standpoint. We consider first how to extend Radford's theorem to the products in that class, then how to construct their bi-algebras. As some conditions are necessary do carry that out, we study them closely and simplify them so that they can be seen directly from the definition of the product. We eventually test these conditions on the products mentioned above.

Keywords: polyz tas functions, combinatorics of φ -shuffle products, comultiplication, Hopf algebra.

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1. Introduction

As a matter of fact, mathematics (in particular number theory), physics and other sciences provide, for their theories, algebras of special functions indexed by parameters¹, with a product, defined at first as a function $X^* \times X^*$ to $A\langle X \rangle$ and satisfying a simple

¹The combinatorial supports of these parameters will finally resolve themselves into words.

recurrence of the type

$$\forall(a, b) \in X^2, \forall(u, v) \in (X^*)^2, \quad au \sqcup_{\varphi} vb = a(u \sqcup_{\varphi} bv) + b(au \sqcup_{\varphi} v) + \varphi(a, b)(u \sqcup_{\varphi} v), \quad (1)$$

the initialization being provided by the fact that 1_{X^*} should be a unit. Of course, we will address the question of the existence of such a product, and will extend it by linearity to $A\langle X \rangle$.

However, recall that these special functions are indexed by parameters but, unfortunately, sometimes do not exist for some of their values : the prototype of this case is the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ for $s = 1$. Nevertheless, if these “functions” are seen formally, one can in many cases², define a product on the indices which governs the effective product on the functions³.

Once the formal identity is obtained, there are many ways to write the divergent quantities as limits of terms which fulfil the same identities (truncated or power series)⁴.

Returning to this family of products, we will use a typology based on examples frequently encountered in the literature as well as new ones that we supply in Section 2.

1. Type I : factor φ comes from a product (possibly with zero) between letters (i.e. $X \cup \{0\}$ is a semigroup).
2. Type II : factor φ comes from the deformation of a semigroup product by a bicharacter.
3. Type III : factor φ comes from the deformation of a semigroup product by a colour factor.
4. Type IV : factor φ is the commutative law of an associative algebra (CAA) on $A.X$
5. Type V : factor φ is the law of an associative algebra (AA) on $A.X$

These classes are ordered by the following (strict) inclusion diagram:

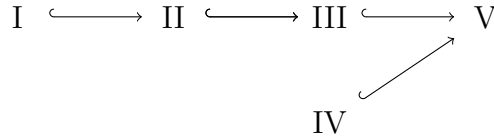


Figure 1: Hasse diagram of the inclusions between classes.

²That includes in particular all the cases under consideration in our paper

³That is the domain of symbolic computation in the vein of Euler and Arbogast[20, 16].

⁴That is the domain of renormalisation and asymptotic analysis initiated by Du Bois-Reymond and Hardy[1, 17].

We have collected examples from the literature, with the corresponding formulas, in the following table.

| Name | Formula (recursion) | φ | Type |
|--------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------|------|
| Shuffle [21] | $au \sqcup bv = a(u \sqcup bv) + b(au \sqcup v)$ | $\varphi \equiv 0$ | I |
| Stuffle [19] | $x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v) + x_{i+j}(u \sqcup v)$ | $\varphi(x_i, x_j) = x_{i+j}$ | I |
| Min-stuffle [7] | $x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v) - x_{i+j}(u \sqcup v)$ | $\varphi(x_i, x_j) = -x_{i+j}$ | III |
| Muffle [14] | $x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v) + x_{i \times j}(u \sqcup v)$ | $\varphi(x_i, x_j) = x_{i \times j}$ | I |
| q -shuffle [3] | $x_i u \sqcup_q x_j v = x_i(u \sqcup_q x_j v) + x_j(x_i u \sqcup_q v) + qx_{i+j}(u \sqcup_q v)$ | $\varphi(x_i, x_j) = qx_{i+j}$ | III |
| q -shuffle ₂ | $x_i u \sqcup_q x_j v = x_i(u \sqcup_q x_j v) + x_j(x_i u \sqcup_q v) + q^{i \cdot j} x_{i+j}(u \sqcup_q v)$ | $\varphi(x_i, x_j) = q^{i \cdot j} x_{i+j}$ | II |
| LDIAG(1, q_s) [10] (non-crossed, non-shifted) | $au \sqcup bv = a(u \sqcup bv) + b(au \sqcup v) + q_s^{ a b } a.b(u \sqcup v)$ | $\varphi(a, b) = q_s^{ a b } (a.b)$ | II |
| q -Infiltration [12] | $au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b} a(u \uparrow v)$ | $\varphi(a, b) = q\delta_{a,b} a$ | III |
| AC-stuffle | $au \sqcup_\varphi bv = a(u \sqcup_\varphi bv) + b(au \sqcup_\varphi v) + \varphi(a, b)(u \sqcup_\varphi v)$ | $\varphi(a, b) = \varphi(b, a)$ $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$ | IV |
| Semigroup-stuffle | $x_t u \sqcup_\perp x_s v = x_t(u \sqcup_\perp x_s v) + x_s(x_t u \sqcup_\perp v) + x_{t \perp s}(u \sqcup_\perp v)$ | $\varphi(x_t, x_s) = x_{t \perp s}$ | I |
| φ -shuffle | $au \sqcup_\varphi bv = a(u \sqcup_\varphi bv) + b(au \sqcup_\varphi v) + \varphi(a, b)(u \sqcup_\varphi v)$ | $\varphi(a, b)$ law of AAU | V |

Of course, the q -shuffle is equal to the (classical) shuffle when $q = 0$. As for the q -infiltration, when $q = 1$, one recovers the infiltration product defined in [6].

Many shuffle products arise in number theory when one studies polylogarithms, harmonic sums and polyzêtas: it was in order to study all these products that two of us introduced Type IV (see above) [15].

On the other hand, in combinatorial physics, one has coproducts with bi-multiplicative (and noncommutative) perturbation factors (see [11]).

The structure of the paper is the following: in part 2, we complete the first products of [15] with the description of products which come from Hurwitz polyzêta functions (the product given in [18] was not valid in all cases) and from generalized Polylerch functions. We are able to give the complete recursive relation which allows to define all kinds of products; we verify that it implies the existence and uniqueness of this product, which can be extended to $A\langle X \rangle$. We examine the “known” and the “new” products in order to determine their classes. In part 3, we consider how to extend Radford’s theorem and we prove that it can be carried over to the whole class of AC-products (class IV): the Lyndon words constitute a pure transcendence basis of the corresponding commutative algebra, which can moreover be endowed, under additional growth conditions, with a Hopf algebra structure.

The basis of Lyndon words is the key to effective computations on the algebra of

special functions ruled by such products⁵. In part 4, we determine the necessary and sufficient conditions on φ so that $\boxtimes \varphi$ belong to the class of AC-products; we give also necessary and sufficient conditions for such a product to be dualizable (*i.e.* to be the adjoint of a comultiplication).

Preliminary remark. It is worth emphasizing at the outset that, although some of the objects/results under review in the present paper have already been defined/proved elsewhere, we include them in our study to lay out as complete a picture as possible and to exemplify the rather 'pedestrian' approach we have adopted. In particular, we have refrained throughout the paper from using more sophisticated algebraic techniques.

Notation. In the sequel, X will denote an alphabet, k a \mathbb{Q} -algebra, and A a k -commutative and associative algebra with unit (a k -CAAU).

2. Hurwitz Polyzêtas and Generalized Polylerch Functions

We remind the reader of some special functions introduced in [14] and complete their study: we prove that they follow a product law which we describe.

2.1. Some special functions and their products

The *Riemann Polyzêta* is the function which maps every composition $\mathbf{s} = (s_1, \dots, s_r) \in (\mathbb{N}_{\geq 1})^r$, to⁶

$$\zeta(\mathbf{s}) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{n_1^{s_1} \dots n_r^{s_r}} \quad (2)$$

We now make an observation which, however simple, will appear in different disguises as a building block of many a construction of the paper : There is a (linear) bijection between the module freely generated by (all) compositions and $\mathbb{Q}\langle Y \rangle$ (where $Y = \{y_k\}_{k \geq 1}$) defined by

$$\beta_s : (s_1, \dots, s_r) \mapsto y_{s_1} \dots y_{s_r} \quad (3)$$

So, if $\mathbf{s} = (s_1, \dots, s_r) \in (\mathbb{N}_{\geq 1})^r$, $s_1 > 1$ and $\mathbf{s}' = (s'_1, \dots, s'_{r'})$, $s'_1 > 1$ are compositions, one knows [14] that⁷

$$\zeta(\mathbf{s} \boxtimes \mathbf{s}') = \zeta(\mathbf{s})\zeta(\mathbf{s}') \quad (4)$$

That function ζ is well-known and is a special case of the following special functions.

⁵The decomposition algorithm (which we shall not describe in detail) is based on formula (36) of lemma (4).

⁶The following series converges for $s_1 > 1$. Under that condition, the definition can be extended by linearity to the module generated by the set of so-called *admissible* composition.

⁷With a slight abuse of language. Stricly speaking, equation 4 actually reads

$$\zeta\left(\beta_s^{-1}(\beta_s(\mathbf{s}) \boxtimes \beta_{s'}(\mathbf{s}'))\right) = \zeta(\mathbf{s})\zeta(\mathbf{s}') .$$

2.1.1. Coloured Polyzêtas

The *coloured polyzêta* is the function which, to a composition $\mathbf{s} = (s_1, \dots, s_r)$ and a tuple of complex numbers of the same length $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r)$, associates

$$\zeta(\mathbf{s}, \boldsymbol{\xi}) = \sum_{n_1 > \dots > n_r > 0} \frac{\xi_1^{n_1} \dots \xi_r^{n_r}}{n_1^{s_1} \dots n_r^{s_r}} \quad (5)$$

It should be noted that $\zeta(\mathbf{s}, \boldsymbol{\xi})$ appears – with the notation $\text{Li}_{\mathbf{s}}(\boldsymbol{\xi})$ – in particule physics [25].

To describe the product here, we will use two alphabets $Y = \{y_i\}_{i \in \mathbb{N}^*}$, $X = \{x_i\}_{i \in \mathbb{C}^*}$ and M be the (free) submonoid generated by $Y \times X$. One easily checks that⁸

$$M = \{(u, v) \in Y^* \times X^* \mid |u| = |v|\}$$

As above, to make things rigorous (but slightly more difficult to read), one considers the (linear) bijection defined, on M , by

$$\beta_c : ((s_1, \dots, s_r), (\xi_1, \dots, \xi_r)) \mapsto (y_{s_1} \dots y_{s_r}, x_{\xi_1} \dots x_{\xi_r}) .$$

The duffle product is defined as follows.

Definition 1 ([15]). (*Product of coloured polyzêtas*) Let $Y = \{y_i\}_{i \in \mathbb{N}^*}$, $X = \{x_i\}_{i \in \mathbb{C}^*}$ and M be as above.

The duffle is defined as a bilinear product over $k[M] = k\langle Y \times X \rangle$ such that

$$\begin{aligned} \forall w \in M^*, \quad w \mathbin{\boxtimes} 1_{M^*} &= 1_{M^*} \mathbin{\boxtimes} w = w, \\ \forall y_i, y_j \in Y^2, \forall x_k, x_l \in X^2, \forall u, v \in M^{*2}, \quad & (y_i, x_k).u \mathbin{\boxtimes} (y_j, x_l).v = (y_i, x_k)(u \mathbin{\boxtimes} (y_j, x_l).v) \\ & + (y_j, x_l)((y_i, x_k).u \mathbin{\boxtimes} v) + (y_{i+j}, x_{k \times l})(u \mathbin{\boxtimes} v). \end{aligned}$$

Again, we will show that, under suitable conditions⁹

$$\zeta((\mathbf{s}, \boldsymbol{\xi}) \mathbin{\boxtimes} (\mathbf{s}', \boldsymbol{\xi}')) = \zeta(\mathbf{s}, \boldsymbol{\xi}) \zeta(\mathbf{s}', \boldsymbol{\xi}') . \quad (6)$$

2.1.2. Hurwitz Polyzêtas

The *Hurwitz polyzêta* is the function which, to a composition $\mathbf{s} = (s_1, \dots, s_r)$ and a tuple of parameters¹⁰ of the same length $\mathbf{t} = (t_1, \dots, t_r)$, associates

$$\zeta(\mathbf{s}, \mathbf{t}) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{(n_1 - t_1)^{s_1} \dots (n_r - t_r)^{s_r}} . \quad (7)$$

⁸Throughout the paper $|w|$ stands for the length of the word w .

⁹Again, rigorously speaking, the left-hand side of the following equation should read

$$\zeta\left(\beta_c^{-1}\left(\beta_c(\mathbf{s}, \boldsymbol{\xi}) \mathbin{\boxtimes} \beta_c(\mathbf{s}', \boldsymbol{\xi}')\right)\right) .$$

¹⁰ All parameters in the tuple are taken in some subring of \mathbb{C} and none of them is a strictly positive integer.

This series converges if and only if $s_1 > 1$ (for a “global” way to expand (7) as a meromorphic function of $\mathbf{s} \in \mathbb{C}^r$, see [13]). To be able to cope with the case $s_1 = 1$, we have to use the *truncated Hurwitz polyzêtas* function given by :

$$\forall N \in \mathbb{N}_{>0}, \quad \zeta_N(\mathbf{s}, \mathbf{t}) = \sum_{N \geq n_r > \dots > n_1 > 0} \frac{1}{(n_1 - t_1)^{s_1} \dots (n_r - t_r)^{s_r}} \quad (8)$$

In order to obtain the product law, we will use here two alphabets $Y = \{y_i\}_{i \in \mathbb{N}_{>0}}$, $Z = \{z_t\}_{t \in \mathbb{C} \setminus \mathbb{N}_{>0}}$, the (free) submonoid N generated by $Y \times Z$ and, as usual, the bijection

$$\beta_h : ((s_1, \dots, s_r), (t_1, \dots, t_r)) \mapsto (y_{s_1} \dots y_{s_r}, z_{t_1} \dots z_{t_r}) \quad (9)$$

suitably extended by linearity. We have now the following product

Definition 2. (*Product of Formal Hurwitz Polyzêtas*) Let $Y = \{y_i\}_{i \in \mathbb{N}^*}$, $Z = \{z_t\}_{t \in k}$ and N be as above.

The *huffle* is defined as a bilinear product over $k[N] = k\langle Y \times Z \rangle$ such that

$$\begin{aligned} \forall w \in N^*, \quad & w \mathbin{\boxplus} 1_{N^*} = 1_{N^*} \mathbin{\boxplus} w = w, \\ \forall y_i, y_j \in Y^2, \forall z_t, z_{t'} \in Z^2, \forall u, v \in N^{*2}, \\ t = t' \Rightarrow & (y_i, z_t)u \mathbin{\boxplus} (y_j, z_t)v \\ & = (y_i, z_t)(u \mathbin{\boxplus} (y_j, z_t)v) + (y_j, z_t)((y_i, z_t)u \mathbin{\boxplus} v) \\ & \quad + (y_{i+j}, z_t)(u \mathbin{\boxplus} v) \\ t \neq t' \Rightarrow & (y_i, z_t).u \mathbin{\boxplus} (y_j, z_{t'}) . v \\ & = (y_i, z_t).(u \mathbin{\boxplus} (y_j, z_{t'}) . v) + (y_j, z_{t'}) . ((y_i, z_t).u \mathbin{\boxplus} v) \\ & \quad + \sum_{n=0}^{i-1} \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} (y_{i-n}, z_t). (u \mathbin{\boxplus} v) \\ & \quad + \sum_{n=0}^{j-1} \binom{i-1+n}{i-1} \frac{(-1)^n}{(t'-t)^{i+n}} (y_{j-n}, z_{t'}) . (u \mathbin{\boxplus} v) . \end{aligned}$$

We also will show that¹¹ for all integer N

$$\zeta_N((\mathbf{s}, \mathbf{t}) \mathbin{\boxplus} (\mathbf{s}', \mathbf{t}')) = \zeta_N(\mathbf{s}, \mathbf{t}) \zeta_N(\mathbf{s}', \mathbf{t}') . \quad (10)$$

Remark 1. The functions we call ‘Hurwitz polyzêtas’, a term coined in the last century (see for example [18]). must not be confused with the monocenter polyzêtas, defined only for a composition \mathbf{s} and a parameter t by

$$\zeta(\mathbf{s}, t) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{(n_1 - t)^{s_1} \dots (n_r - t)^{s_r}}, \quad (11)$$

which follow a much simpler rule, namely the *stuffle* product on the compositions.

¹¹Again, rigorously speaking, the left-hand side of the following equation should read

$$\zeta_N \left(\beta_h^{-1} \left(\beta_h(\mathbf{s}, \mathbf{t}) \mathbin{\boxplus} \beta_h(\mathbf{s}', \mathbf{t}') \right) \right) .$$

2.1.3. Generalized Polylêrch functions

The *generalized Polylêrch function* is the function which maps a composition $\mathbf{s} = (s_1, \dots, s_r)$, a tuple $\xi = (\xi_1, \dots, \xi_r)$ of complex numbers, and a tuple $\mathbf{t} = (t_1, \dots, t_r)$ of parameters¹⁰, all three of the same length, to

$$\zeta(\mathbf{s}, \mathbf{t}, \xi) = \sum_{n_1 > \dots > n_r > 0} \frac{\xi_1^{n_1} \dots \xi_r^{n_r}}{(n_1 - t_1)^{s_1} \dots (n_r - t_r)^{s_r}}. \quad (12)$$

Here, we will need three alphabets $Y = \{y_i\}_{i \in \mathbb{N}^*}$, $X = \{x_i\}_{i \in \mathbb{C}^*}$, $Z = \{z_t\}_{t \in k}$ and the (free) submonoid T generated by $Y \times Z \times X$. The bijection

$$\beta_l : ((s_1, \dots, s_r), (t_1, \dots, t_r), (\xi_1, \dots, \xi_r)) \mapsto (y_{s_1} \dots y_{s_r}, z_{t_1} \dots z_{t_r}, x_{\xi_1} \dots x_{\xi_r}) \quad (13)$$

still extended by linearity. The product \boxtimes is given by the following definition:

Definition 3. *Product of Generalized Lerch functions*

Let $Y = \{y_i\}_{i \in \mathbb{N}^*}$, $X = \{x_i\}_{i \in \mathbb{C}^*}$, $Z = \{z_t\}_{t \in k}$ and T be the (free) submonoid generated by $Y \times Z \times X$.

The luffle is defined as the bilinear product over $k[T] = k\langle Y \times Z \times X \rangle$ satisfying the following recursive relation :

$$\begin{aligned} \forall w \in A^*, \quad w \boxtimes 1_{A^*} &= 1_{A^*} \boxtimes w = w, \\ \forall (y_i, y_j) \in Y^2, \forall (z_t, z_{t'}) \in Z^2, \forall (x_k, x_l) \in X^2, \forall (u, v) \in A^{*2}, \\ t = t' &\Rightarrow (y_i, z_t, x_k).u \boxtimes (y_j, z_t, x_l).v \\ &= (y_i, z_t, x_k).(u \boxtimes (y_j, z_t).v) + (y_j, z_t, x_l).((y_i, z_t).u \boxtimes v) \\ &\quad + (y_{i+j}, z_t, x_{k \times l}).(u \boxtimes v) \\ t \neq t' &\Rightarrow (y_i, z_t, x_k).u \boxtimes (y_j, z_{t'}, x_l).v \\ &= (y_i, z_t, x_k).(u \boxtimes (y_j, z_{t'}).v) + (y_j, z_{t'}, x_l).((y_i, z_t).u \boxtimes v) \\ &\quad + \sum_{n=0}^{i-1} \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} (y_{i-n}, z_t, x_{k \times l}).(u \boxtimes v) \\ &\quad + \sum_{n=0}^{j-1} \binom{i-1+n}{i-1} \frac{(-1)^n}{(t'-t)^{i+n}} (y_{j-n}, z_{t'}, x_{k \times l}).(u \boxtimes v) \end{aligned}$$

We also show¹²

$$\zeta((\mathbf{s}, \mathbf{t}, \xi) \boxtimes (\mathbf{s}', \mathbf{t}', \xi')) = \zeta(\mathbf{s}, \mathbf{t}, \xi) \zeta(\mathbf{s}', \mathbf{t}', \xi'). \quad (15)$$

2.2. General framework of study

Other products from table 1 belong to the same family as the examples examined so far, and so pertain to the same kind of approach. As we aim to offer as comprehensive a framework as possible, we now concentrate on the most general class of φ -products, *i.e.* class V, which emerges from definition (4) below. We will use a unitary ring as the ground set of scalars (and not a field as it would be expected in combinatorics) because some applications require to work with rings of (analytic or arithmetic) functions.

¹²Again, rigorously speaking, the left-hand side of equation 15 should read

$$\zeta \left(\beta_l^{-1} \left(\beta_l(\mathbf{s}, \mathbf{t}, \xi) \boxtimes \beta_l(\mathbf{s}', \mathbf{t}', \xi') \right) \right). \quad (14)$$

Proposition 1. *Let A be a unitary commutative ring, X be an alphabet and $\varphi : X \times X \rightarrow A\langle X \rangle$ is an arbitrary mapping. Then there exists a unique mapping $\star : X^* \times X^* \rightarrow A\langle X \rangle$ satisfying the conditions :*

$$(R) \begin{cases} \text{for any } w \in X^*, 1_{X^*} \star w = w \star 1_{X^*} = w, \\ \text{for any } a, b \in X \text{ and } u, v \in X^*, \\ au \star bv = a(u \star bv) + b(au \star v) + \varphi(a, b)(u \star v). \end{cases} \quad (16)$$

Proof — By recurrence over $n = |u| + |v|$. \square

Definition 4. *With the notations of Proposition 1, the unique mapping from $X \times X$ to $A\langle X \rangle$ satisfying conditions (R) will be noted \sqcup_φ and will be called φ -shuffle product.*

From now on, we suppose that φ takes its values in AX the space of homogeneous polynomials of degree 1. We still denote by φ its linear extension to $AX \otimes AX$ given by

$$\varphi(P, Q) = \sum_{x, y \in X} \langle P|x \rangle \langle Q|y \rangle \varphi(x, y) \quad (17)$$

and \sqcup_φ the extension of the mapping of Definition (4) by linearity¹³ to $A\langle X \rangle \otimes A\langle X \rangle$. Then \sqcup_φ becomes a law of algebra (with 1_{X^*} as unit) on $A\langle X \rangle$.

2.3. Extending quasi-stuffle relations

The following elementary result can be found in any complex analysis textbook. It is freely used throughout this section.

Lemma 1. *For any integers $s, r \geq 1$, for any complex numbers $a, b \neq a$:*

$$\forall x \in \mathbb{C} \setminus \{a, b\}, \frac{1}{(x-a)^s(x-b)^r} = \sum_{k=1}^s \frac{a_k}{(x-a)^k} + \sum_{k=1}^r \frac{b_k}{(x-b)^k} \quad (18)$$

where, for all $k \in \{1, \dots, s\}$, $a_k = \binom{s+r-k-1}{r-1} \frac{(-1)^{s-k}}{(a-b)^{s+r-k}}$

and, for all $k \in \{1, \dots, r\}$, $b_k = \binom{s+r-k-1}{s-1} \frac{(-1)^{r-k}}{(b-a)^{s+r-k}}$.

Let $\mathbf{t} = (t_1, \dots, t_r)$ be a set of parameters¹⁰, $\mathbf{s} = (s_1, \dots, s_r)$ a composition, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r) \in \mathbb{C}^r$. We define, for $N \in \mathbb{N}_{>0}$,

$$M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N = \sum_{N \geq n_1 > \dots > n_r > 0} \prod_{i=1}^r \frac{\xi_i^{n_i}}{(n_i - t_i)^{s_i}}. \quad (19)$$

and $M_{(0,0),0}^N = 1$.

Of course, it is a truncated series of $\zeta(\mathbf{s}; \mathbf{t}; \boldsymbol{\xi})$.

¹³We recall that AX (resp. $A\langle X \rangle$) admits X (resp. X^*) as linear basis, therefore $AX \otimes AX$ (resp. $A\langle X \rangle \otimes A\langle X \rangle$) is free with basis $X \times X$ (resp. $X^* \times X^*$) or more precisely, the image family $(x \otimes y)_{x, y \in X}$ (resp. $(u \otimes v)_{u, v \in X^*}$).

Proposition 2. For every composition \mathbf{s} , tuple $\boldsymbol{\xi}$ of complex numbers, tuple \mathbf{t} of parameters all of the same length $l \in \mathbb{N}$, and for every composition \mathbf{r} , tuple $\boldsymbol{\rho}$ of complex numbers, tuple \mathbf{t}' of parameters also of the same length $k \in \mathbb{N}$, one has

$$\forall N \in \mathbb{N}, \quad M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N = M_{(\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}) \sqcup (\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}')}^N. \quad (20)$$

Proof — If $l = 0$ or $k = 0$, that is immediate.

Let $l \in \mathbb{N}^*$, $k \in \mathbb{N}^*$ and $\mathbf{s} = (s_1, \dots, s_l)$ and $\mathbf{r} = (r_1, \dots, r_k)$ two compositions, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_l) \in \mathbb{C}^l$, $\boldsymbol{\rho} = (\rho_1, \dots, \rho_k) \in \mathbb{C}^k$, and $\mathbf{t} = (t_1, \dots, t_l)$, $\mathbf{t}' = (t'_1, \dots, t'_k)$ two sets of parameters and put $\mathbf{s}_2 = (s_2, \dots, s_l)$, $\mathbf{r}_2 = (r_2, \dots, r_k)$, $\boldsymbol{\xi}_2 = (\xi_2, \dots, \xi_l)$, $\boldsymbol{\rho}_2 = (\rho_2, \dots, \rho_k)$, $\mathbf{t}_2 = (t_2, \dots, t_l)$ and $\mathbf{t}'_2 = (t'_2, \dots, t'_k)$,

- If $t'_1 = t_1$,

$$M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}}^N = \sum_{N \geq n_1, N \geq n'_1} \frac{\xi_1^{n_1}}{(n_1 - t_1)^{s_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^{n_1} \frac{\rho_1^{n'_1}}{(n'_1 - t_1)^{r_1}} M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}_2}^{n'_1} \quad (21)$$

Classically, we decompose the sum $\sum_{N \geq n_1, N \geq n'_1}$ into three sums corresponding to the simplices $n_1 > n'_1$; $n'_1 > n_1$ and $n_1 = n'_1$ and get

$$\begin{aligned} M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}}^N &= \sum_{N \geq n_1} \frac{\xi_1^{n_1}}{(n_1 - t_1)^{s_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^{n_1} M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}}^{n_1} + \sum_{N \geq n'_1} \frac{\rho_1^{n'_1}}{(n'_1 - t'_1)^{r_1}} M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^{n'_1} M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}_2}^{n'_1} \\ &+ \sum_{N \geq m} \frac{(\xi_1 \rho_1)^m}{(m - t_1)^{s_1}} \frac{1}{(m - t'_1)^{r_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^m M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}_2}^m \end{aligned} \quad (22)$$

so that,

$$\forall N \in \mathbb{N}, \quad M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}}^N = M_{(\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}) \sqcup (\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}')}^N. \quad (23)$$

- In the same way, when $t_1 \neq t'_1$

$$\begin{aligned} M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N &= \sum_{N \geq n_1} \frac{\xi_1^{n_1}}{(n_1 - t_1)^{s_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^{n_1} M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^{n_1} + \sum_{N \geq n'_1} \frac{\rho_1^{n'_1}}{(n'_1 - t'_1)^{r_1}} M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^{n'_1} M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}_2}^{n'_1} \\ &+ \sum_{N \geq m} (\xi_1 \rho_1)^m \frac{1}{(m - t_1)^{s_1}} \frac{1}{(m - t'_1)^{r_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^m M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}_2}^m \\ &= \sum_{N \geq n_1} \frac{\xi_1^{n_1}}{(n_1 - t_1)^{s_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^{n_1} M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^{n_1} + \sum_{N \geq n'_1} \frac{\rho_1^{n'_1}}{(n'_1 - t'_1)^{r_1}} M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^{n'_1} M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}_2}^{n'_1} \\ &+ \sum_{N \geq m} \left[\sum_{k=1}^{s_1} \binom{s_1 + r_1 - k - 1}{r_1 - 1} \frac{(-1)^{s_1 - k}}{(t_1 - t'_1)^{s_1 + r_1 - k}} \frac{(\xi_1 \rho_1)^m}{(m - t'_1)^k} \right. \\ &\left. + \sum_{k=1}^{r_1} \binom{s_1 + r_1 - k - 1}{s_1 - 1} \frac{(-1)^{r_1 - k}}{(t_1 - t'_1)^{s_1 + r_1 - k}} \right] M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^m M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}_2}^m \end{aligned} \quad (24)$$

so

$$\forall N \in \mathbb{N}, \quad M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N = M_{(\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}) \sqcup (\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}')}^N. \quad (25)$$

□

Remark 2. Let r a integer, $\chi = (\chi_1, \dots, \chi_r)$ a tuple of multiplicative characters¹⁴ and $(\mathbf{s}, \boldsymbol{\xi}, \mathbf{t})$ being as above) let us define

$$M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N(\chi) = \sum_{N \geq n_1 > \dots > n_r > 0} \prod_{i=1}^r \frac{\chi_i^{n_i}(\xi_i)}{(n_i - t_i)^{s_i}}. \quad (26)$$

The same proof shows that, for any $(\mathbf{s}, \boldsymbol{\xi}) \in \mathbb{Z}_{>0}^l \times \mathbb{C}^l$ and $(\mathbf{r}, \boldsymbol{\rho}) \in \mathbb{Z}_{>0}^k \times \mathbb{C}^k$, for any l -tuple \mathbf{t} and k -tuple \mathbf{t}' of parameters¹⁰,

$$\forall N \in \mathbb{N}, \quad M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N(\chi) M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N(\chi) = M_{(\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}) \boxplus (\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}')}^N(\chi). \quad (27)$$

This result allows to deduce some product relations on the different multi-zêta functions :

Theorem 2.1. Let $\mathbf{s} = (s_1, \dots, s_l)$ and $\mathbf{r} = (r_1, \dots, r_k)$ two compositions, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_l)$ a l -tuple, $\boldsymbol{\rho} = (\rho_1, \dots, \rho_k)$ a k -tuple of complex numbers of which the first composant has a modulus strictly less than 1, $\mathbf{t} = (t_1, \dots, t_s)$ and $\mathbf{t}' = (t'_1, \dots, t'_k)$ two tuples of parameters not in $\mathbb{N}_{>0}$ and $N \in \mathbb{N}$

(i) For the coloured polyzêta function :

$$\zeta(\mathbf{s}, \boldsymbol{\xi}) \zeta(\mathbf{s}', \boldsymbol{\xi}') = \zeta((\mathbf{s}, \boldsymbol{\xi}) \boxplus (\mathbf{s}', \boldsymbol{\xi}')) \quad (28)$$

(ii) For the truncated Hurwitz polyzêta function :

$$\zeta_N(\mathbf{s}, \mathbf{t}) \zeta_N(\mathbf{s}', \mathbf{t}') = \zeta_N((\mathbf{s}, \mathbf{t}) \boxplus (\mathbf{s}', \mathbf{t}')) \quad (29)$$

(iii) In particular, for the monocentered polyzêta function :

$$\zeta(\mathbf{s}, (t, \dots, t)) \zeta(\mathbf{s}', (t, \dots, t)) = \zeta((\mathbf{s}, (t, \dots, t)) \boxplus (\mathbf{s}', (t, \dots, t))) \quad (30)$$

where t is a parameter s.t. $t \notin \mathbb{N}_{>0}$.

(iv) For the Polylerch generalized function :

$$\zeta(\mathbf{s}, \mathbf{t}, \boldsymbol{\xi}) \zeta(\mathbf{s}', \mathbf{t}', \boldsymbol{\xi}') = \zeta((\mathbf{s}, \mathbf{t}, \boldsymbol{\xi}) \boxplus (\mathbf{s}', \mathbf{t}', \boldsymbol{\xi}')) \quad (31)$$

Proof — (ii) comes directly from Proposition 2 because $\zeta_N(\mathbf{s}, \mathbf{t}) = M_{\mathbf{s}, (1, \dots, 1), \mathbf{t}}^N$; for (i), (iii) and (iv), apply Proposition 2 with, respectively, the functions

$$M_{\mathbf{s}, \boldsymbol{\xi}, (0, \dots, 0)}^N, \quad M_{\mathbf{s}, (1, \dots, 1), (t, \dots, t)}^N \quad \text{and} \quad M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N$$

and take both sides of the equality to the limit as N grows to infinity. \square

¹⁴Endomorphisms of the semigroup (\mathbb{C}, \times) .

Remark 3. We cannot use this method for the Hurwitz polyzêtas because in the decomposition, some divergent terms (which have $s_1 = 1$!) appear: for example, for $t \neq t'$,

$$\begin{aligned}
(y_2, z_t) \sqcup (y_3, z_{t'}) &= (y_2 y_3, z_t z_{t'}) + (y_3 y_2, z_{t'} z_t) + \sum_{n=0}^1 \binom{2+n}{2} \frac{(-1)^n}{(t-t')^{3+n}} (y_{2-n}, z_t) \\
&\quad + \sum_{n=0}^2 \binom{1+n}{1} \frac{(-1)^n}{(t-t')^{2+n}} (y_{3-n}, z_{t'}) \\
&= (y_2 y_3, z_t z_{t'}) + (y_3 y_2, z_{t'} z_t) + \frac{1}{(t-t')^3} (y_2, z_t) - \frac{3}{(t-t')^4} (y_1, z_t) \\
&\quad + \frac{1}{(t-t')^2} (y_3, z_{t'}) - \frac{2}{(t-t')^3} (y_1, z_{t'}) + \frac{3}{(t-t')^4} (y_1, z_{t'}) \quad (32)
\end{aligned}$$

Separately, the terms $-\frac{3}{(t-t')^4} (y_1, z_t)$ and $\frac{3}{(t-t')^4} (y_1, z_{t'})$, corresponding respectively to $\frac{-3}{(t-t')^4} \frac{1}{n-t}$ and $\frac{3}{(t-t')^4} \frac{1}{n-t'}$ give a divergent series although all other terms correspond to convergent series. Of course, the sum of the two

$$\frac{3}{(t-t')^4} \left(-\frac{1}{n-t} + \frac{1}{n-t'} \right) = \frac{3}{(t-t')^4} \left(\frac{t'-t}{(n-t)(n-t')} \right) \quad (33)$$

is a term of a convergence series, but the series is not a Hurwitz Polyzêta.

3. Radford's theorem for the AC-stuffle.

In this subsection, A is supposed to be a ring with unit; when we need it to be commutative or to contain the set of rational numbers, we will state it explicitly.

Let $<$ be a total ordering on the alphabet X , and $\mathcal{Lyn}(X)$ denote the family of Lyndon words [24] constructed from X^* w.r.t. this ordering. We will prove that the largest framework in which Radford's theorem holds true [23] is when φ is commutative (and associative).

3.1. Computing φ -shuffle expressions using shuffles

In this subsection A is a ring with unit and $\varphi : AX \otimes AX \rightarrow AX$ an associative law. We can express the result of the φ -shuffle product thanks to the shuffle product (and some terms of lower degree). First we observe what happens with the product of two words :

Lemma 2. For $u, v \in X^*$, there exists $(C_{u,v}^w)_{|w| < |u|+|v|} \in A^{(\mathbb{N})}$ such that :

$$u \sqcup_{\varphi} v = u \sqcup v + \sum_{|w| < |u|+|v|} C_{u,v}^w w.$$

Proof — Omitted. □

Now, because the Lyndon words are candidates to be a transcendental basis, we see what happens when they are φ -shuffled.

Definition 5. Let $\star : A\langle X \rangle \times A\langle X \rangle \mapsto A\langle X \rangle$ be an associative law with unit and $\mathbb{X} = \text{Lyn}(X)$. For any $\alpha \in \mathbb{N}^{(\mathbb{X})}$ and $\{l_1, \dots, l_r\} \supset \text{supp}(\alpha)$ in strict decreasing order (i.e. $l_1 > \dots > l_r$), we set

$$\mathbb{X}^{\star\alpha} = l_1^{\star\alpha_1} \star \dots \star l_r^{\star\alpha_r} \quad (34)$$

One easily checks easily that the product (34) does not depend on the choice of $\{l_1, \dots, l_r\} \supset \text{supp}(\alpha)$. We will also need the following parameter (which will turn out to be the length of the dominant terms in the product)

$$||\alpha|| = \sum_{l \in \text{Lyn}(X)} \alpha(l) |l|. \quad (35)$$

Lemma 3. If \sqcup_{φ} is associative,

$$\forall \alpha \in \mathbb{N}^{(\text{Lyn}(X))}, \exists (C_{\beta}^{\alpha})_{\beta} \in A^{(\mathbb{N}^{(\text{Lyn}(X))})} / \mathbb{X}^{\sqcup_{\varphi} \alpha} = \mathbb{X}^{\sqcup \alpha} + \sum_{\substack{\beta \in \mathbb{N}^{(\text{Lyn}(X))} \\ ||\beta|| < ||\alpha||}} C_{\beta}^{\alpha} \mathbb{X}^{\sqcup \beta}.$$

Proof — Omitted □

3.2. Radford's theorem in φ -shuffle algebras

Lemma 4. If \sqcup_{φ} is associative,

$$\forall p \in \mathbb{N}^*, \text{span}((\mathbb{X}^{\sqcup_{\varphi} \alpha})_{\alpha \in \mathbb{N}^{(\text{Lyn}(X))}, ||\alpha|| < p}) = \text{span}((\mathbb{X}^{\sqcup \alpha})_{\alpha \in \mathbb{N}^{(\text{Lyn}(X))}, ||\alpha|| < p}). \quad (36)$$

Proof — Lemma 3 give $\forall p \in \mathbb{N}^*, \text{span}((\mathbb{X}^{\sqcup_{\varphi} \alpha})_{\alpha \in \mathbb{N}^{(\text{Lyn}(X))}, ||\alpha|| < p}) \subset \text{span}((\mathbb{X}^{\sqcup \alpha})_{\alpha \in \mathbb{N}^{(\text{Lyn}(X))}, ||\alpha|| < p})$. We just have to prove, for any $p \in \mathbb{N}^*$, the property $\mathcal{P}(p)$:

$$\text{span}((\mathbb{X}^{\sqcup \alpha})_{\alpha \in \mathbb{N}^{(\text{Lyn}(X))}, ||\alpha|| < p}) \subset \text{span}((\mathbb{X}^{\sqcup_{\varphi} \alpha})_{\alpha \in \mathbb{N}^{(\text{Lyn}(X))}, ||\alpha|| < p}) \quad (37)$$

- It is true for $p = 1$.

- Assume $\mathcal{P}(p)$ true for an integer p .

Let $\alpha \in \mathbb{N}^{(\text{Lyn}(X))}$ such that $||\alpha|| < p + 1$.

We can find $(C_{\beta}^{\alpha})_{\beta} \in A^{(\mathbb{N}^{(\text{Lyn}(X))})}$ such that $\mathbb{X}^{\sqcup_{\varphi} \alpha} = \mathbb{X}^{\sqcup \alpha} + \sum_{\substack{\beta \in \mathbb{N}^{(\text{Lyn}(X))} \\ ||\beta|| < ||\alpha||}} C_{\beta}^{\alpha} \mathbb{X}^{\sqcup \beta}$, so

$$\mathbb{X}^{\sqcup \alpha} = \mathbb{X}^{\sqcup_{\varphi} \alpha} - \sum_{\substack{\beta \in \mathbb{N}^{(\text{Lyn}(X))} \\ ||\beta|| < ||\alpha||}} C_{\beta}^{\alpha} \mathbb{X}^{\sqcup \beta}.$$

But every term of the sum is of the form $C_{\beta}^{\alpha} \mathbb{X}^{\sqcup \beta}$ with $\beta \in \mathbb{N}^{(\text{Lyn}(X))}$ and $||\beta|| < ||\alpha|| < p + 1$ so $||\beta|| < p$.

Consequently, they are in $\text{span}((\mathbb{X}^{\sqcup_{\varphi} \alpha})_{\alpha \in \mathbb{N}^{(\text{Lyn}(X))}, ||\alpha|| < p})$, and so is the sum. By the induction hypothesis, the sum is in $\text{span}((\mathbb{X}^{\sqcup_{\varphi} \alpha})_{\alpha \in \mathbb{N}^{(\text{Lyn}(X))}, ||\alpha|| < p})$, therefore $\mathbb{X}^{\sqcup \alpha} \in \text{span}((\mathbb{X}^{\sqcup_{\varphi} \alpha})_{\alpha \in \mathbb{N}^{(\text{Lyn}(X))}, ||\alpha|| < p+1})$. □

Theorem 3.1. *Let A be a commutative ring (with unit) such that $\mathbb{Q} \subset A^{15}$ and $\sqcup_\varphi : A\langle X \rangle \otimes A\langle X \rangle \rightarrow A\langle X \rangle$ is associative. If X is totally ordered by $<$, then $(\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}(\mathcal{L}^{yn}(X))}$ is a linear basis of $A\langle X \rangle$.*

Proof — Since this family is a generating family by lemma 4, only freeness remains to be proven.

Let $\sum_{\alpha \in J} \beta_\alpha \mathbb{X}^{\sqcup_\varphi \alpha} = 0$ be a null linear combination of $(\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}(\mathcal{L}^{yn}(X))}$, with J a nonempty finite subset of $\mathbb{N}(\mathcal{L}^{yn}(X))$. Thanks to lemma 3, for any $\alpha \in J$, we can find a finite family $B_\alpha \subset \mathbb{N}(\mathcal{L}^{yn}(X))$ and $(C_\beta^\alpha)_{\beta \in B_\alpha} \in A^{B_\alpha}$ such that

$$\mathbb{X}^{\sqcup_\varphi \alpha} = \mathbb{X}^{\sqcup \alpha} + \sum_{\substack{\beta \in B_\alpha \\ \|\beta\| < \|\alpha\|}} C_\beta^\alpha \mathbb{X}^{\sqcup \beta}.$$

Set $B = J \cup \left(\bigcup_{\alpha \in J} B_\alpha \right)$; B is a finite set. Then $(\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in J}$ is a triangular family for $|\cdot|$ with respect to the family $\mathcal{F} = (\mathbb{X}^{\sqcup \alpha})_{\alpha \in B}$ in the vector space $\text{span}(\mathcal{F})$, which is of finite dimension. But \mathcal{F} is a basis, so $(\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in J}$ is free and $\forall \alpha \in J, \beta_\alpha = 0$. \square

Corollary 1. *Under the same hypotheses, if in addition \sqcup_φ is commutative in A then*

- i) *The algebra $\mathcal{A} = (A\langle X \rangle, \sqcup_\varphi, 1_{X^*})$ is a polynomial algebra.*
- ii) *$\text{Lyn}(X)$ is a transcendence basis of \mathcal{A} .*

Remark 4. *It is necessary to suppose $\mathbb{Q} \subset A$ as, in case $\varphi \equiv 0$, one has*

$$\forall n \in \mathbb{N}_{>0}, a^n = \frac{1}{n!} (a^{\sqcup n}) \quad (38)$$

Proof —

- i) Immediate result.
- ii) Comes from proposition 3.1 and theorem 4.1, which proves in an elementary (so independent) way that the commutativity of φ is equivalent to the commutativity of \sqcup_φ .

\square

¹⁵Precisely, $\mathbb{N}^+.1_A \subset A^\times$

3.3. Bialgebra structure

Definition 6. A law \star defined over $A\langle X \rangle$ is a dual law (or dualizable) if there exists a linear mapping $\Delta_\star : A\langle X \rangle \rightarrow A\langle X \rangle \otimes A\langle X \rangle$ such

$$\forall (u, v, w) \in X^* \times X^* \times X^*, \quad \langle u \star v | w \rangle = \langle u \otimes v | \Delta_\star(w) \rangle^{\otimes 2}. \quad (39)$$

In this case, Δ_\star will be called the comultiplication dual to \star .

Theorem 3.2. If A is a commutative ring (with unit), if $\mathbb{Q} \subset A$, and if in addition the product $\sqcup_\varphi : A\langle X \rangle \otimes A\langle X \rangle \rightarrow A\langle X \rangle$ is an associative and commutative law on $A\langle X \rangle$, then the algebra $(A\langle X \rangle, \sqcup_\varphi, 1_{X^*})$ can be endowed with the comultiplication Δ_{conc} dual to the concatenation

$$\Delta_{\text{conc}}(w) = \sum_{uv=w} u \otimes v \quad (40)$$

and the “constant term” character $\epsilon(P) = \langle P | 1_{X^*} \rangle$.

With this setting

$$\mathcal{B}_\varphi = (A\langle X \rangle, \sqcup_\varphi, 1_{X^*}, \Delta_{\text{conc}}, \epsilon) \quad (41)$$

is a bialgebra¹⁶.

Remark 5. Let, classically, Δ_{conc}^+ be defined by

$$\forall w \in X^*, \Delta_{\text{conc}}^+(w) = \sum_{\substack{uv=w \\ u, v \neq 1}} u \otimes v$$

We remark that Δ_{conc}^+ is coassociative and locally nilpotent, i.e.

$$\forall w \in X^*, \exists n \in \mathbb{N}^* / (\Delta_{\text{conc}}^+)^n(w) = 0.$$

Thus the bialgebra (41) is, in fact, a Hopf Algebra.

Proof — It is a classical combinatorial verification, done in [15].

The following identity remains to be proven:

$$\forall (w_1, w_2) \in X^*, \Delta_{\text{conc}}(w_1 \sqcup_\varphi w_2) = \Delta_{\text{conc}}(w_1) \Delta_{\text{conc}}(w_2) \quad (42)$$

which can be done by a (lengthy) induction or by duality. \square

4. Conditions for AC-shuffle and dualizability

4.1. Commutative and associative conditions

We have obtained an extended version of Radford’s theorem and other properties with conditions stated w.r.t. \sqcup_φ , we will see in this subsection that these conditions can be set uniquely in terms of properties of φ itself.

¹⁶Commutative and, when $|X| \geq 2$, noncocommutative.

Definition 7. For $P \in A\langle X \rangle$, we note $\text{supp}(P)$ the support of P and $\deg(P) = \max\{|l|, l \in \text{supp}(P)\}$

Lemma 5. Let A be a unitary commutative ring, X be an alphabet and $\varphi : X \times X \rightarrow A\langle X \rangle$ is an arbitrary mapping.
Then,

$$\forall (u, v) \in (X^*)^2, \deg(u \sqcup_{\varphi} v) \leq |u| + |v| \quad (43)$$

Proof — If $|u| = 0$ or $|v| = 0$, then $u \sqcup_{\varphi} v$ is one of $\{u, v\}$ so its length is $|u| + |v|$.
Let X^+ be the set of nonempty words. We prove $\forall (u, v) \in (X^+)^2, \deg(u \sqcup_{\varphi} v) = |u| + |v|$ by induction on $|u| + |v|$.
For any letters a and b , $a \sqcup_{\varphi} b = ab + ba + \varphi(a, b)1_{A^*}$ so $\deg(a \sqcup_{\varphi} b) = 2 = |a| + |b|$.
One assumes the property true for all words $u, v \in X^+$ such that $|u| + |v| = n$, where n is an integer. Let u and v be now two words of X^+ such that $|u| + |v| = n + 1$.
There exist x, y in X , u', v' in X^* such that $u = xu', v = yv'$ (because $(u, v) \in (X^+)^2$).
Then $|u| + |v'| = |u| + |v| - 1 \leq n$, so $\deg(y(u \sqcup_{\varphi} v')) \leq n + 1$. Also $|u'| + |v| = |u| - 1 + |v| \leq n$ so $\deg(x(u' \sqcup_{\varphi} v)) \leq n + 1$, and $|u'| + |v'| = |u| - 1 + |v| - 1 \leq n$ so $\deg(\varphi(x, y)u' \sqcup_{\varphi} v') \leq n + 1$.
Hence, $\deg(u \sqcup_{\varphi} v) = n + 1$: the induction is proved. \square

Theorem 4.1. In the context of definition 4,

- (i) The law \sqcup_{φ} is commutative if and only if the extension $\varphi : AX \otimes AX \rightarrow AX$ is commutative.
- (ii) The law \sqcup_{φ} is associative if and only if the extension $\varphi : AX \otimes AX \rightarrow AX$ is associative.

Proof — We give an elementary proof.

- (i) $[\sqcup_{\varphi} \text{ commutative} \implies \varphi \text{ commutative}]$

Let us suppose $\forall (u, v) \in (X^*)^2, u \sqcup_{\varphi} v = v \sqcup_{\varphi} u$.

In particular, $\forall (x, y) \in (X^*)^2, x \sqcup_{\varphi} y = x \sqcup_{\varphi} y$. But, for any $(x, y) \in X^2$,

$$x \sqcup_{\varphi} y = xy + yx + \varphi(x, y) \quad \text{and} \quad y \sqcup_{\varphi} x = yx + xy + \varphi(y, x). \quad (44)$$

and so $(\forall x, y \in X)(\varphi(x, y) = \varphi(y, x))$.

$[\varphi \text{ commutative} \implies \sqcup_{\varphi} \text{ commutative}]$

Now let us suppose φ is commutative then let us prove by recurrence on $|uv|$ that \sqcup_{φ} is commutative :

- The previous equivalence proves that the recurrence holds for $|u| = |v| = 1$.
- Suppose the recurrence holds for any $u, v \in X^*$ such that $2 \leq |uv| \leq n$ and $|u|, |v| \neq 1$.

Let $u = xu'$ and $v = yv'$ with $x, y \in X$ and $u', v' \in X^*$. Then,

$$\begin{aligned} u \sqcup_{\varphi} v &= x(u' \sqcup_{\varphi} yv') + y(xu' \sqcup_{\varphi} v) + \varphi(x, y)(u' \sqcup_{\varphi} v') \\ &= x(yv \sqcup_{\varphi} u) + y(v' \sqcup_{\varphi} xu') + \varphi(y, x)(v' \sqcup_{\varphi} u') \quad (\text{by the induction hypothesis}) \\ &= v \sqcup_{\varphi} u. \end{aligned} \quad (45)$$

(ii) $[\sqcup_\varphi \text{ associative} \implies \varphi \text{ associative}]$ Let us suppose

$$\forall u, v, w \in X^*, \quad (u \sqcup_\varphi v) \sqcup_\varphi w = u \sqcup_\varphi (v \sqcup_\varphi w). \quad (46)$$

Then, for any $x, y, z \in X$, one has

$$(x \sqcup_\varphi y) \sqcup_\varphi z = x \sqcup_\varphi (y \sqcup_\varphi z). \quad (47)$$

But

$$\begin{aligned} (x \sqcup_\varphi y) \sqcup_\varphi z &= (xy + yx + \varphi(x, y)) \sqcup_\varphi z \\ &= xy \sqcup_\varphi z + yx \sqcup_\varphi z + \varphi(x, y) \sqcup_\varphi z \\ &= x(y \sqcup_\varphi z) + z(xy \sqcup_\varphi 1) + \varphi(x, z)y + y(x \sqcup_\varphi z) + z(yx \sqcup_\varphi 1) + \varphi(y, z)x \\ &\quad + \varphi(x, y)z + z\varphi(x, y) + \varphi(\varphi(x, y), z) \\ &= x(yz + zy + \varphi(y, z)) + zxy + \varphi(x, z)y + y(xz + zx + \varphi(x, z)) + zyx \\ &\quad + \varphi(y, z)x + \varphi(x, y)z + z\varphi(x, y) + \varphi(\varphi(x, y), z) \end{aligned} \quad (48)$$

$$\begin{aligned} x \sqcup_\varphi (y \sqcup_\varphi z) &= x \sqcup_\varphi (yz + zy + \varphi(y, z)) \\ &= x \sqcup_\varphi yz + x \sqcup_\varphi zy + x \sqcup_\varphi \varphi(y, z) \\ &= x(1 \sqcup_\varphi yz) + y(x \sqcup_\varphi z) + \varphi(x, y)z + x(1 \sqcup_\varphi zy) + z(x \sqcup_\varphi y) + \varphi(x, z)y \\ &= x\varphi(y, z) + \varphi(y, z)x + \varphi(x, \varphi(y, z)) \\ &= xyz + y(xz + zx + \varphi(x, z)) + \varphi(x, y)z + xzy + z(xy + yx + \varphi(x, y)) + \varphi(x, z)y \\ &\quad + x\varphi(y, z) + \varphi(y, z)x + \varphi(x, \varphi(y, z)). \end{aligned} \quad (49)$$

One can then deduce that

$$(\forall x, y, z \in X)(x \sqcup_\varphi (y \sqcup_\varphi z) = (x \sqcup_\varphi y) \sqcup_\varphi z) \iff (\forall x, y, z \in X)(\varphi(x, \varphi(y, z)) = \varphi(\varphi(x, y), z)). \quad (51)$$

$[\varphi \text{ associative} \implies \sqcup_\varphi \text{ associative}]$ Now if φ is associative then let us prove by induction on $|u| + |v| + |w|$ that \sqcup_φ is associative :

- The previous equivalence proves that the induction holds for $|u| = |v| = |w| = 1$.
- Suppose the recurrence holds for any $u, v \in X^*$ such that $3 \leq |u| + |v| + |w| \leq n$ and $|u|, |v|, |w| \neq 1$.
- Let $u = xu', v = yv'$ and $w = zw'$ with $x, y, z \in X$ and $u', v', w' \in X^*$. Then,

$$\begin{aligned} &u \sqcup_\varphi (v \sqcup_\varphi w) \\ &= u \sqcup_\varphi (y(v' \sqcup_\varphi w) + z(v \sqcup_\varphi w') + \varphi(y, z)(v' \sqcup_\varphi w')) \\ &= x(u' \sqcup_\varphi y(v' \sqcup_\varphi w)) + y(u \sqcup_\varphi (v' \sqcup_\varphi w)) + \varphi(x, y)(u' \sqcup_\varphi (v' \sqcup_\varphi w)) \\ &\quad + x(u' \sqcup_\varphi z(v \sqcup_\varphi w')) + z(u \sqcup_\varphi (v \sqcup_\varphi w')) + \varphi(x, z)(u' \sqcup_\varphi (v \sqcup_\varphi w')) \\ &\quad + x(u' \sqcup_\varphi \varphi(y, z)(v' \sqcup_\varphi w')) + \varphi(y, z)(u \sqcup_\varphi (v' \sqcup_\varphi w')) + \varphi(x, \varphi(y, z))u' \sqcup_\varphi (v' \sqcup_\varphi w') \\ &= x(u' \sqcup_\varphi (v \sqcup_\varphi w)) \\ &\quad + y(u \sqcup_\varphi (v' \sqcup_\varphi w)) + \varphi(x, y)(u' \sqcup_\varphi (v' \sqcup_\varphi w)) \\ &\quad + z(u \sqcup_\varphi (v \sqcup_\varphi w')) + \varphi(x, z)(u' \sqcup_\varphi (v \sqcup_\varphi w')) \\ &\quad + \varphi(y, z)(u \sqcup_\varphi (v' \sqcup_\varphi w')) + \varphi(x, \varphi(y, z))u' \sqcup_\varphi (v' \sqcup_\varphi w') \end{aligned} \quad (52)$$

and

$$\begin{aligned}
& (u \sqcup_{\varphi} v) \sqcup_{\varphi} w \\
&= (x(u' \sqcup_{\varphi} v) + y(u \sqcup_{\varphi} v') + \varphi(x, y)(u' \sqcup_{\varphi} v')) \sqcup_{\varphi} w \\
&= x((u' \sqcup_{\varphi} v) \sqcup_{\varphi} w) + z(x(u' \sqcup_{\varphi} v) \sqcup_{\varphi} w') + \varphi(x, z)((u' \sqcup_{\varphi} v) \sqcup_{\varphi} w') \\
&+ y((u \sqcup_{\varphi} v') \sqcup_{\varphi} w) + z(y(u \sqcup_{\varphi} v') \sqcup_{\varphi} w') + \varphi(y, z)((u \sqcup_{\varphi} v') \sqcup_{\varphi} w') \\
&+ \varphi(x, y)((u' \sqcup_{\varphi} v') \sqcup_{\varphi} w) + z(\varphi(x, y)(u' \sqcup_{\varphi} v') \sqcup_{\varphi} w') + \varphi(\varphi(x, y), z)((u' \sqcup_{\varphi} v') \sqcup_{\varphi} w') \\
&= x((u' \sqcup_{\varphi} v) \sqcup_{\varphi} w) + \varphi(x, z)((u' \sqcup_{\varphi} v) \sqcup_{\varphi} w') \\
&+ y((u \sqcup_{\varphi} v') \sqcup_{\varphi} w) + \varphi(y, z)((u \sqcup_{\varphi} v') \sqcup_{\varphi} w') \\
&+ \varphi(x, y)((u' \sqcup_{\varphi} v') \sqcup_{\varphi} w) + \varphi(\varphi(x, y), z)((u' \sqcup_{\varphi} v') \sqcup_{\varphi} w') \\
&+ z(u \sqcup_{\varphi} v) \sqcup_{\varphi} w'
\end{aligned} \tag{53}$$

Indeed, thanks to the induction hypothesis and the commutativity of φ , $u \sqcup_{\varphi} (v \sqcup_{\varphi} w)$ and $(u \sqcup_{\varphi} v) \sqcup_{\varphi} w$ are equal.

□

4.2. Dualizability conditions

Proposition 3. We call $\gamma_{x,y}^z := \langle \varphi(x, y) | z \rangle$ the structure constants of φ (w.r.t. the basis X).

The product \sqcup_{φ} is a dual law if and only if $(\gamma_{x,y}^z)_{x,y,z \in X}$ is dualizable in the following sense

$$(\forall z \in X)(\#\{(x, y) \in X^2 | \gamma_{x,y}^z \neq 0\} < +\infty) . \tag{54}$$

Proof — $[\sqcup_{\varphi} \text{ dual law} \implies \gamma_{x,y}^z \text{ dualizable}]$. Let Δ be the dual of \sqcup_{φ} , that is, for all $u, v, w \in X^*$

$$\langle u \sqcup_{\varphi} v | w \rangle = \langle u \otimes v | \Delta(w) \rangle^{\otimes 2} . \tag{55}$$

For all $z \in X$, one must have $\Delta(z) = \sum_{i=1}^n \alpha_i u_i \otimes v_i$. On the other hand, for all $x, y \in X$, one has $(x \sqcup_{\varphi} y) - (xy + yx) = \varphi(x, y)$. Hence

$$\begin{aligned}
\gamma_{x,y}^z &= \langle \varphi(x, y) | z \rangle = \langle (x \sqcup_{\varphi} y) - (xy + yx) | z \rangle = \langle (x \sqcup_{\varphi} y) | z \rangle - \langle (xy + yx) | z \rangle \\
&= \langle (x \otimes y) | \Delta(z) \rangle = \langle (x \otimes y) | \sum_{i=1}^n \alpha_i u_i \otimes v_i \rangle .
\end{aligned} \tag{56}$$

We can deduce from the preceding argument that

$$\gamma_{x,y}^z \neq 0 \implies (x \in \cup_{i=1}^n \text{Alph}(u_i) \text{ and } y \in \cup_{i=1}^n \text{Alph}(v_i))$$

which proves the point.

$[\gamma_{x,y}^z \text{ dualizable} \implies \sqcup_{\varphi} \text{ dual law}]$. This is, combinatorially speaking, the most interesting point. We first define a comultiplication Δ on $A\langle X \rangle$ by transposing the structure constants of \sqcup_{φ} by

$$\Delta(z) := z \otimes 1 + 1 \otimes z + \sum_{x,y \in X} \gamma_{x,y}^z x \otimes y \tag{57}$$

and, as the sum is finite (see however the comment after this theorem), this quantity belongs to $A\langle X \rangle \otimes A\langle X \rangle$. One then has a linear mapping $\Delta : AX \rightarrow A\langle X \rangle \otimes A\langle X \rangle$ which is extended, by universal property, into a morphism of algebras $\Delta : A\langle X \rangle \rightarrow A\langle X \rangle \otimes A\langle X \rangle$. Explicitely, for all $w = z_1 z_2 \cdots z_n$, one has

$$\Delta(z_1 z_2 \cdots z_n) = \Delta(z_1) \Delta(z_2) \cdots \Delta(z_n) . \tag{58}$$

Now, we prove that the dual law of the latter coproduct is exactly \sqcup_φ .
First remark : by (57) and (58), one has

$$\Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v \in X^+} \beta(u,v) u \otimes v \quad (59)$$

the last sum being finitely supported. This shows by duality that

$$u \sqcup_\Delta 1 = 1 \sqcup_\Delta u = u \quad (60)$$

(here, \sqcup_Δ stands for the dual law of Δ). Moreover

$$\begin{aligned} au \sqcup_\Delta bv &= \sum_{w \in X^*} \langle au \sqcup_\Delta bv | w \rangle w \\ &= \sum_{w \in X^*} \langle au \otimes bv | \Delta(w) \rangle w \\ &= \langle au \otimes bv | 1 \otimes 1 \rangle 1 + \sum_{w \in X^+} \langle au \otimes bv | \Delta(w) \rangle w \\ &= \sum_{x \in X; m \in X^*} \langle au \otimes bv | \Delta(xm) \rangle xm \\ &= \sum_{x \in X; m \in X^*} \langle au \otimes bv | \Delta(x) \Delta(m) \rangle xm \\ &= \sum_{x \in X; m \in X^*} \langle au \otimes bv | \left(x \otimes 1 + 1 \otimes x + \sum_{y,z \in X} \langle \Delta(x) | y \otimes z \rangle y \otimes z \right) \Delta(m) \rangle xm \\ &= \sum_{x \in X; m \in X^*} \langle au \otimes bv | (x \otimes 1) \Delta(m) \rangle xm + \sum_{x \in X; m \in X^*} \langle au \otimes bv | (1 \otimes x) \Delta(m) \rangle xm + \\ &\quad \sum_{x \in X; m \in X^*} \langle au \otimes bv | \sum_{y,z \in X} \langle \Delta(x) | y \otimes z \rangle y \otimes z \Delta(m) \rangle xm \\ &\quad \sum_{m \in X^*} \langle au \otimes bv | (a \otimes 1) \Delta(m) \rangle am + \sum_{m \in X^*} \langle au \otimes bv | (1 \otimes b) \Delta(m) \rangle bm + \\ &\quad \sum_{x \in X; m \in X^*} \langle au \otimes bv | \langle \Delta(x) | a \otimes b \rangle a \otimes b \Delta(m) \rangle xm \\ &\quad \sum_{m \in X^*} \langle u \otimes bv | \Delta(m) \rangle am + \sum_{m \in X^*} \langle au \otimes v | \Delta(m) \rangle bm + \\ &\quad \sum_{x \in X; m \in X^*} \langle \Delta(x) | a \otimes b \rangle \langle u \otimes v | \Delta(m) \rangle xm \\ &= a \sum_{m \in X^*} \langle u \otimes bv | \Delta(m) \rangle m + b \sum_{m \in X^*} \langle au \otimes v | \Delta(m) \rangle m + \\ &\quad \sum_{m \in X^*} \left(\sum_{x \in X} \langle \Delta(x) | a \otimes b \rangle x \right) \langle u \otimes v | \Delta(m) \rangle m \\ &= a(u \sqcup_\Delta bv) + b(au \sqcup_\Delta v) + \varphi(a,b)(u \sqcup_\Delta v) \end{aligned} \quad (61)$$

This proves that the dual law \sqcup_Δ equals \sqcup_φ and we are done. \square

4.3. The Hopf-Hurwitz algebra

In section 7, we provided the law on indices followed by the product of Formal Hurwitz polyzêtas, we now prove that the law φ associated with it is associative. The “centres” will be taken from a subfield k of \mathbb{C} and the set of coefficients A is a k -CAAU.

Proposition 4. i) The law $\varphi : AN \otimes AN \rightarrow AN$ associated to \circ is defined, on the basis N , by the multiplication table $\mathcal{T}_{\text{Formal Hurwitz}}$

$$\begin{aligned}
& \text{if } t = t'; \quad \varphi((y_i, z_t), (y_j, z_{t'})) = (y_{i+j}, z_t) \\
& \text{if } t \neq t'; \quad \varphi((y_i, z_t), (y_j, z_{t'})) = \\
& \quad \sum_{n=0}^{i-1} \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} (y_{i-n}, z_t) + \sum_{n=0}^{j-1} \binom{i-1+n}{i-1} \frac{(-1)^n}{(t'-t)^{i+n}} (y_{j-n}, z_{t'})
\end{aligned}
\tag{62}$$

ii) The product \boxplus is associative, commutative and unital, making $(A\langle N \rangle, \circ, 1_N)$ into a A -CAAU.

Proof — i) Let first $j : kN \rightarrow k(X)$ be the linear mapping defined by $j((y_i, z_t)) = \frac{1}{(X-t)^i}$. In fact, as the $\{\frac{1}{(X-t)^i}\}$ are linearly independent, j is into. On the other hand, j is a morphism of k -AAU due to the fact that the multiplication table is identical. Hence φ is a law of \mathbb{C} -AA on $AN \simeq A \otimes_k kN$. ii) Is a consequence of the general theorems. \square

Now, we have the following bialgebra

$$\mathcal{H}_{\text{Formal Hurwitz}} = (A\langle N \rangle, \boxplus_\varphi, 1_{N^*}, \Delta_{\text{conc}}, \epsilon) \tag{63}$$

which is a Hopf algebra. Note that \boxplus_φ is not dualisable which means that the adjoint

$$\Delta_{\boxplus_\varphi} : N^* \rightarrow A\langle\langle N^* \otimes N^* \rangle\rangle \tag{64}$$

does not have its image in $A\langle N \rangle \otimes A\langle N \rangle$. See next paragraph for tools and proofs.

Corollary 2. The product \boxplus is associative, commutative and unital, making $(A\langle N \rangle, \boxplus, 1_N)$ into a A -CAAU.

Proof — It comes that the product \boxplus is a direct product of the products \circ and \boxdot . \square

5. Conclusion

We have been able to give a useful extended version of Radford's theorem. Let us observe that :

- For the shuffle product, $\varphi_{\sqcup} \equiv 0$, so the shuffle \sqcup is associative, commutative and dualizable.
- The stuffle product over an alphabet indexed by \mathbb{N} is associative and commutative because $\varphi_{\boxplus}(x_i, x_j) = x_{i+j}$ is so; moreover it is dualizable.
- The muffle product over an alphabet indexed by \mathbb{C} is associative and commutative because $\varphi_{\boxdot}(x_i, x_j) = x_{i \times j}$ is so; it is not dualizable because for all $n \in \mathbb{N}_{>0}$, $x_1 = \varphi_{\boxdot}(x_{1/n}, x_n)$.
However, there are multiplicative subsemigroups S of \mathbb{C} such that φ restricted to the alphabet $(x_i)_{i \in S}$ is dualizable.

- The duffle product over an alphabet indexed by $\mathbb{N}^* \times \mathbb{C}^*$ is associative and commutative because $\varphi_{\blacksquare}((y_i, x_k), (y_j, x_l)) = (y_{i+j}, x_{k+l})$ is associative and commutative; it is not dualizable either (for the same reason).
But we can do the same remark as the muffle about the possibility to restrict the alphabet so that φ becomes dualizable.
- The Formal Polyzêta product is associative, commutative but, for all $j \in \mathbb{N}_{>0}$, with $t \neq t'$,

$$\begin{aligned} \varphi_{\circ}((y_2, z_t), (y_j, z_{t'})) &= \sum_{n=0}^1 \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} (y_{2-n}, z_t) \\ &\quad + \sum_{n=0}^{j-1} \binom{1+n}{1} \frac{(-1)^n}{(t'-t)^{2+n}} (y_{j-n}, z_{t'}) \end{aligned} \quad (65)$$

and then $\forall j \in \mathbb{N}_{>0}, \quad \langle \varphi_{\circ}((y_2, z_t), (y_j, z_{t'})) | (y_1, z_t) \rangle = -\binom{j}{j-1} \frac{1}{(t-t')^{j+1}} \neq 0$
which implies that \circ is not dualizable.

- The Lerch product is associative, commutative and not dualizable (for the same reason as \blacksquare).

So, if we work in the Riemann polyzêta algebra, in the coloured polyzêta algebra, or in the Generalized Lerch polyzêta algebra, we can use a representation with the Lyndon set as a transcendence basis. Moreover, in the Riemann polyzêta algebra and the truncated Hurwitz polyzêta algebra can both be completed into Hopf algebras.

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